Winning ways for your mathematical plays

by Aart Blokhuis

Winning Ways for your Mathematical Plays by Elwyn R. Berlekamp, John H. Conway & Richard K. Guy.
Vol. 1 Games in General, 426 pp.
Vol. 2 Games in Particular, 424 pp.
Academic Press (London, New York, etc.), 1982.

One-heap min is certainly the most boring game ! Two persons sit at a table with a heap of beans. At his turn one of the players takes any number of beans, at least one, from the heap. The first player unable to move loses. Usually the first player wins by taking the whole heap. Things become more interesting, though, if more heaps are involved. At his turn a player chooses a heap and removes any number from that heap. This game is the *disjunctive sum* of several one-heap min games. In general, games are considered in which the players move alternately, and with the rule that the first player unable to move loses. In the disjunctive sum of games, a move consists of choosing a game and making a move in that game. In Volume I of Winning Ways, the authors develop a theory for addition of games, best illustrated using a partial variant of min called hackenbush.

A hackenbush game is drawn in figure I.

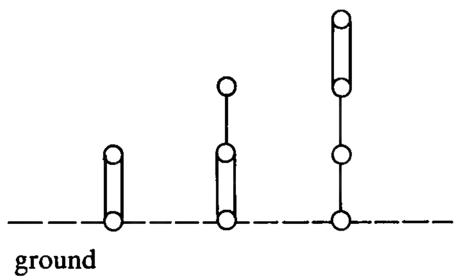


Fig.1

The two players, blue and red say, may hack the bush in any segment of their colour (single edges = blue, double = red). The part of the stalk that is disconnected from the ground disappears, together with the chopped segment. Clearly, this example is the disjunctive sum of three 'simple' hackenbush games. A little analysis also shows that in this example the player who starts always loses. This is called a *zero-game.* In hackenbush it is possible to assign to every position a number, corresponding to the number of 'free moves' for blue. If the number is positive blue always wins and if it is negative red wins (irrespective of who starts); zero means that the starter loses. In figure 1 the three parts have values $-1, -\frac{1}{2}$ and $1\frac{1}{2}$, adding up to 0. How to compute these numbers is all in the book. As an example here are strings with value π and $1/\omega$, where ω is the first infinite ordinal:

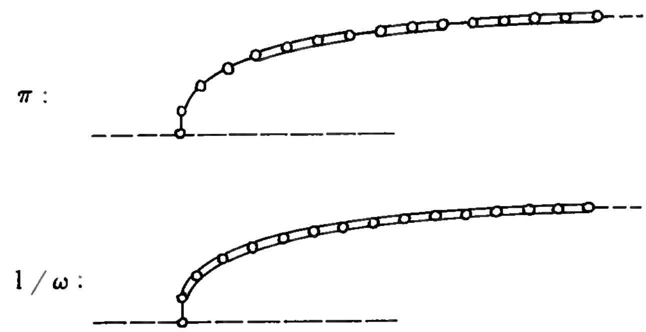


Fig.2

Also, other startling values like $\sqrt{\omega}$ or $\omega^{-\omega}$ can be constructed.

So far things were easy, but what number should we assign to a min-heap of one bean, or (which is the same) to a hackenbush stalk having just one segment which is red and blue at the same time (purple say) ? In this game the first player wins, but if you add to it any game with positive value, i.e. blue wins, the result is still positive. Hence the value of this game is something like zero, but it's not zero itself (which means the starter loses). In this way the 'nimbers' make their appearance. This one is called $\star$, and it is easy to prove that $\star + \star = 0$! Another interesting 'number' is $\uparrow$ which has the property $\uparrow > 0$, but $\uparrow < 2^{-n}$ for all n. It is here that the real problems begin, and that the reader should read the book instead of this review.

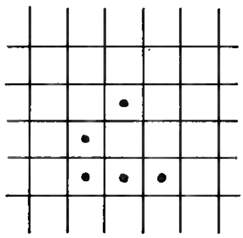
Volume 2 is called *games in particular,* and the only way to give an impression of its contents is to look at some games in particular. In chapter 18 we meet the following game: A and B choose a number in turn, with the restriction that no new number may be the sum of multiples of previously chosen ones. If you choose 1 you lose. For example, A:2, B:5, A:3, B:1 and loses.

The game is called Sylvester coinage. Now B, of course, did not play very smartly, since choosing 3 instead of 5 would have won immediately. Hence 2 was not a clever choice for A either. Less obvious is that 4, 6, 8, 9, 12 are all losing opening moves. Can the reader prove that p is a winning opening choice for each prime $p \geqslant 5$? And what about 16, 18, 24, or any number of the form $2^a 3^b$?

The game Sylvester coinage can possibly be analyzed completely; however the following two-person game (which cannot) is really interesting. It is called 'philosopher's football', or Phutball. The ball, a black stone, is put on the central point of a go-board. Each player, when he moves, *either* places a white stone somewhere on the board, *or* jumps the ball over a series of white stones in any of the eight directions any number of times, removing the jumped-over stones. One side of the board is Left's goal line, the opposite is Right's. Left wins if he succeeds in bringing the ball on (or behind) Right's goal line and conversely. For a little introduction, see p. 688.

Further games analyzed include tic-tac-toe, hare and hounds, fox and geese and many other less well-known games.

After 700 pages on two-person games come an additional 120 on 1-person-games or puzzles, including of course the celebrated Hungarian cube and the game of solitaire. The final 30 pages are about the most interesting no-person game: *life*. *Life* is a 'game' which is played on an infinite chess-board. At every stage some squares are 'alive' and others 'dead'. In the next stage squares become alive or die according to the following rules: A square is born if exactly 3 of its neighbours were alive in the last stage. A square dies if more than 3 or less than 2 neighbours were alive in the previous stage. As an exercise one should look at the development of the 'most spectacular small living object, the glider'. Black dots are living cells.



the glider

Fig.3

After its 'discovery' by J.H. Conway around 1970 *life* has become very popular, as a result of which many amazing starting configurations have been found, whose names suggest their properties: spaceships, flip-flops and finally Gosper's *glider-gun,* emitting a new glider every 30 generations. The final pages of Volume 2 are devoted to a construction of a computer using glider guns, eaters and other ingenious configurations, thus proving that *life is universal.*